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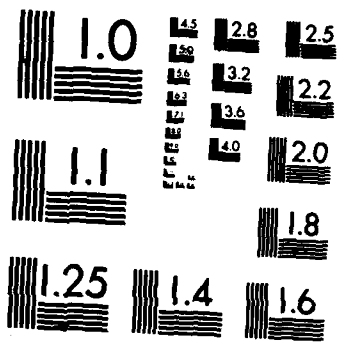

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Monterey, California

OPTIMAL TRIANGULATION
IN THE
PRESENCE OF RANGE ERRORS

by

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This paper is essentially a subset of the report NPS-53-84-0008 by I.B. Russak and A.A. Goldstein with modifications made to the proof of the optimality of the three dimensional triangulation.
The problem of determining the location of a fixed point using range only data occurs in a variety of areas including military applications. An example is outlined below and formed the incentive for the work herein. The application consisted of precisely locating hydrophones placed on a ship's hull for the

BLOCK 20 (CONTINUATION) - ABSTRACT

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OPTIMAL TRIANGULATION

IN THE PRESENCE OF RANGE

ERRORS

A. A. GOLDSTEIN

I. BERT RUSSAK

ABSTRACT

The problem of determining the location of a fixed point using range only data occurs in a variety of areas including military applications. An example is outlined below and formed the incentive for the work herein. The application consisted of precisely locating hydrophones placed on a ship's hull for the purpose of receiving acoustic signals to be used in calculating torpedo positions during a test. In this paper a method is developed which under the assumption of fixed system error input (in range data) produces the smallest possible error in locating the hydrophones.

ANALYSIS

~~Our~~ The goal is to determine the coordinates of a point in 3-dimensional space. For the purpose of simplicity ~~we~~ this document first investigates the 2-dimensional version of this problem and then extends it to the 3-dimensional case.

The determination of the coordinates of a point in the plane, requires a minimum of two observations. We investigate the geometry of that situation. Let positions 1 and 2 be the known observer positions with d the distance between them while p is the hydrophone position which is to be determined. We assume a coordinate system with 1 at the origin and with x-axis defined by 1 and 2. The associated figure is below in which r_1 and r_2 are the respective ranges from observers 1 and 2 to p and θ, α are the indicated angles.

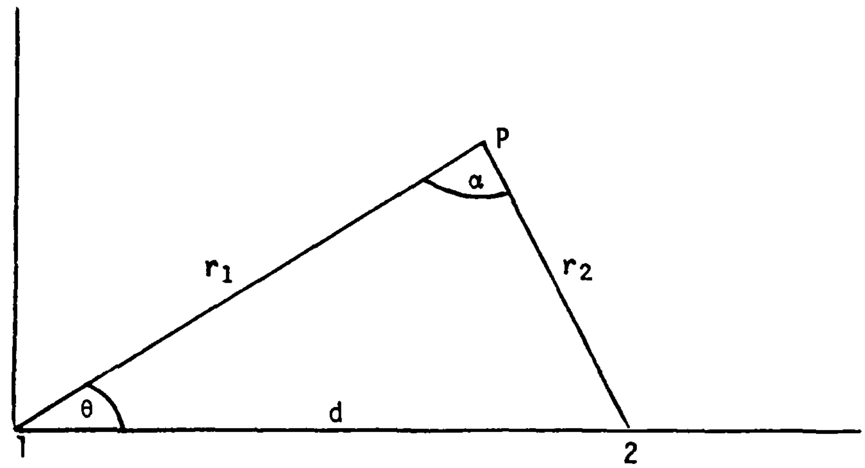

Figure 1

Two Dimensional Nominal Triangularization

The figure is redrawn below with indicated errors Δr_1, Δr_2, $\Delta\theta$ respectively in the ranges r_1, r_2 and in the angle θ . An associated new location point P' determined from the ranges with errors is also drawn.

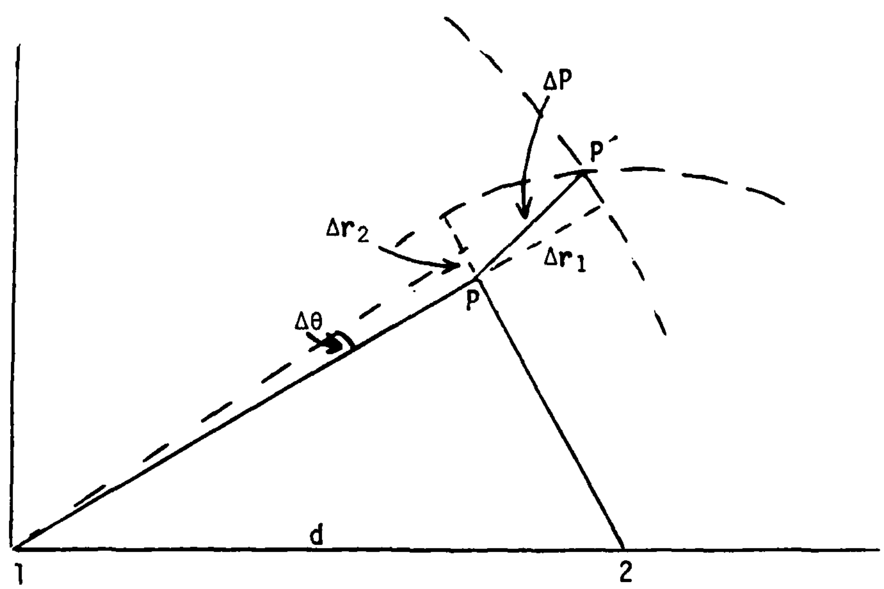

Figure 2

Two Dimensional Triangularization with Errors in the Ranges

The vector difference between positions P and P′ is indicated by $\Delta\vec{P}$ and is seen to be made up of two components, namely the component Δr_1 along r_1 and the component $r_1\Delta\theta$ perpendicular to r_1. Thus with $1\vec{r}_1$ and $1\vec{\theta}$ as unit vectors along r_1 and perpendicular to it respectively, we have the vector equation:

1) $$\Delta\vec{P} = r_1\Delta\theta 1\vec{\theta} + \Delta r_1 1\vec{r}_1$$

Next, according to the law of cosines, we obtain 2a) and 2b) while 2c) is justified by the law of sines.

2a) $$\cos\alpha = \frac{r_1^2 + r_2^2 - d^2}{2r_1r_2}$$

2b) $$\cos\theta = \frac{r_1^2 + d^2 - r_2^2}{2r_1d}$$

2c) $$\frac{r_2}{d} = \frac{\sin\theta}{\sin\alpha}$$

Forming the total differential of 2b), simplifying and solving for $\Delta\theta$ gives

3) $$\Delta\theta = -\frac{1}{r_1d\sin\theta}\left[\frac{r_1^2 + r_2^2 - d^2}{2r_1}\Delta r_1 - r_2\Delta r_2\right]$$

Next, by using 2a) we get

4) $$\Delta\theta = -\frac{r_2}{r_1d\sin\theta}\left[\cos\alpha\,\Delta r_1 - \Delta r_2\right]$$

and by 2c) after simplifying

5) $$\Delta\theta = -\frac{1}{r_1\sin\alpha}\left[\cos\alpha\,\Delta r_1 - \Delta r_2\right]$$

so that after multiplying by r_1

6) $r_1 \Delta\theta = \dfrac{\Delta r_2 - \Delta r_1 \cos \alpha}{\sin \alpha}$

Using absolute values to consider the worst case situations, i.e. where the contributions to $r_1\Delta\theta$ from Δr_1 and Δr_2 add up, we get

7) $|r_1 \Delta\theta| \leq \dfrac{|\Delta r_1 \cos \alpha| + |\Delta r_2|}{\sin \alpha}$

It is clear that 7) that for given errors Δr_1, Δr_2 then the minimum of the worst case situation for $r_1\Delta\theta$ and hence for $||\Delta\vec{P}||$ occurs when $\alpha = \Pi/2$, i.e. when the lines of sight from 1 and 2 to P are perpendicular.

Extension to the 3-Dimensional Case

The previous analysis suggests the following rule for obtaining the minimum error in locating the point P in 3-dimensional space. For a given set of range errors from the observers, then in the worst case, the resulting vector error in locating the point P will be smallest in magnitude if the respective lines of sight from the observers are mutually perpendicular.

This conclusion is also born out by the following alternate analysis:

Let x,y,z be the unknown coordinates of the point P and x_{sk}, y_{sk}, z_{sk} the coordinates of the k^{th} observer and r_k the range from the k^{th} observer to P. Consider the system of equations

8) $\sqrt{(x-x_{sk})^2 + (y-y_{sk})^2 + (z-z_{sk})^2} = r_k \qquad k=1,2,3$

The procedure used to solve 8) is detailed later but is based on forming an approximate solution x_0, y_0, z_0 to 8) and making a correction δx, δy, δz to x_0, y_0, z_0. That correction is the solution of the linear equations

$$9)\quad \frac{(x_0-x_{sk})\delta x + (y_0-y_{sk})\delta y + (z_0-z_{sk})\delta z}{r_{k,0}} = \delta r_k \qquad k = 1,2,3$$

where: $r_{k,0} = \sqrt{(x_0-x_{sk})^2 + (y_0-y_{sk})^2 + (z_0-z_{sk})^2}$ and $\delta r_k = (r_k - r_{k,0})$ i.e. the residual in solving 8) by the approximate solution.

Writing M as the matrix

$$10)\quad M = \begin{bmatrix} \frac{x_0 - x_{s1}}{r_{1,0}} & \frac{y_0 - y_{s1}}{r_{1,0}} & \frac{z_0 - z_{s1}}{r_{1,0}} \\ \frac{x_0 - x_{s2}}{r_{2,0}} & \frac{y_0 - y_{s2}}{r_{2,0}} & \frac{z_0 - z_{s2}}{r_{2,0}} \\ \frac{x_0 - x_{s3}}{r_{3,0}} & \frac{y_0 - y_{s3}}{r_{3,0}} & \frac{z_0 - z_{s3}}{r_{3,0}} \end{bmatrix}$$

then 9) is the system of linear equations

$$11)\quad M \begin{bmatrix} \delta x \\ \delta y \\ \delta z \end{bmatrix} = \begin{bmatrix} \delta r_1 \\ \delta r_2 \\ \delta r_3 \end{bmatrix}$$

If r_k is in error by Δr_k then also δr_k is in error by $\Delta(\delta r_k) = \Delta r_k$. Define the vectors

12a) $\Delta(\delta\vec{r}) = (\Delta(\delta r_1), \Delta(\delta r_2), \Delta(\delta r_3))$

12b) $\Delta\vec{p}$ $(\Delta(\delta x), \Delta(\delta y), \Delta(\delta z))$

so that in the Euclidean norm

12c) $||\Delta(\vec{\delta} r)|| = \sqrt{[\Delta(\delta r_1)]^2 + [\Delta(\delta r_2)]^2 + [\Delta(\delta r_3)]^2}$

12d) $||\Delta\vec{p}|| = \sqrt{[\Delta(\delta x)]^2 + [\Delta(\delta y)]^2 + [\Delta(\delta z)]^2}$

Then referring to 11), it is well known that for a given set of $\Delta(\delta r_k)$ $k = 1,2,3,$ then $||\Delta\vec{P}||$ will be smaller, the closer is the condition number of M to one (see [1]). By examining the rows of M, one sees that these rows are (except for errors in the matrix terms), unit vectors along the lines of sight from observers to P. Thus if these lines of sight are made mutually perpendicular then the matrix M will be an orthogonal matrix and 11) then represents an orthogonal transformation between $(\Delta\vec{P})$ and $\Delta(\delta\vec{r})$. For such a transformation the condition number is equal to one.

Thus for the triangulation problem in 3-dimensional space, a given set of Δr_1, Δr_2, Δr_3, will produce the smallest $||\Delta\vec{P}||$ when the lines of sight from the observers form a mutually orthogonal set of vectors.

Such a situation is accomplished in the following figure in which the three observers are in the x-y plane and point P is below the x-y plane.

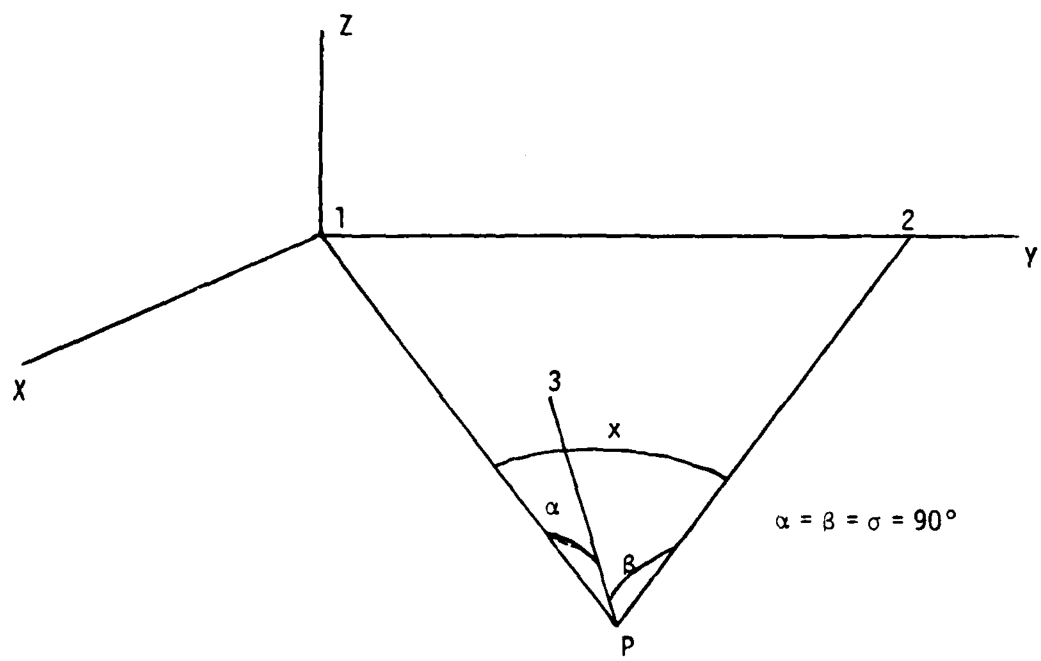

Figure 3

Optimal Observer - Phone Geometry

Method of Solution

The method of solution used is linearized least squares and is now described.

Let $x^{(0)}$ $y^{(0)}$ $z^{(0)}$ be an initial estimate (the method of determining this will be described) of phone position.

We seek to determine hydrophone coordinates x,y,z such that the following equations are satisfied

$$13)\quad \sqrt{(x-x_{sk})^2 + (y-y_{sk})^2 + (z-z_{sk})^2} = r_k \qquad k=1,\ldots,m$$

where m is the number of observations and r_k is the observed range from the $k^{\underline{th}}$ observer.

Let $x^{(i)}$, $y^{(i)}$, $z^{(i)}$ be the estimates of phone coordinates after i iterations and let $r_k^{(i)}$ be the associated range (computed from $x^{(i)}$, $y^{(i)}$, $z^{(i)}$ to observer k. Thus

$$14)\quad \sqrt{(x^{(i)}-x_{sk})^2 + (y^{(i)}-y_{sk})^2 + (z^{(i)}-z_{sk})^2} = r_k^{(i)} \qquad k=1,\ldots,m$$

The correction δx, δy, δz to $x^{(i)}$, $y^{(i)}$, $z^{(i)}$ is computed as the least squares solution of the system.

$$15)\quad \frac{(x^{(i)}-x_{sk})\delta x + (y^{(i)}-y_{sk})\delta y + (z^{(i)}-z_{sk})\delta z}{r_k^{(i)}} = r_k - r_k^{(i)} \qquad k=1,\ldots,m$$

The (i+1)st estimate of phone position is then

$$16)\quad x^{(i+1)} = x^{(i)} + \delta x \qquad y^{(i+1)} = y^{(i)} + \delta y \qquad z^{(i+1)} = z^{(i)} + \delta z$$

and then equations 15) and 16) are re-solved with i+1 replacing i. Convergence of the process is accepted when the change in phone position is less than 10^{-7} on successive estimates.

Initial Solution Point for Least Squares Technique

An initial solution point is necessary in order to use the linearized least squares iterative technique described previously. The equations for determining that initial point now follow:

The figure which indicates the 3-dimensional triangularization is shown below with the observers indicated by 1, 2, 3, and the unknown phone by P. The x-y plane will generally coincide with the surface of the water. For ease of illustration, P has been drawn above the x-y plane when in reality it will be below that plane. However, this distinction will be taken care of by the mathematics to follow. In addition: a) for purpose of simplifying the mathematics, the y axis will be defined as passing through observers 1 and 2 with the origin at observer 1 and; b) the angles u, i are as indicated with i representing the inclination of the plane containing 1, 2 and P to the x-y plane.

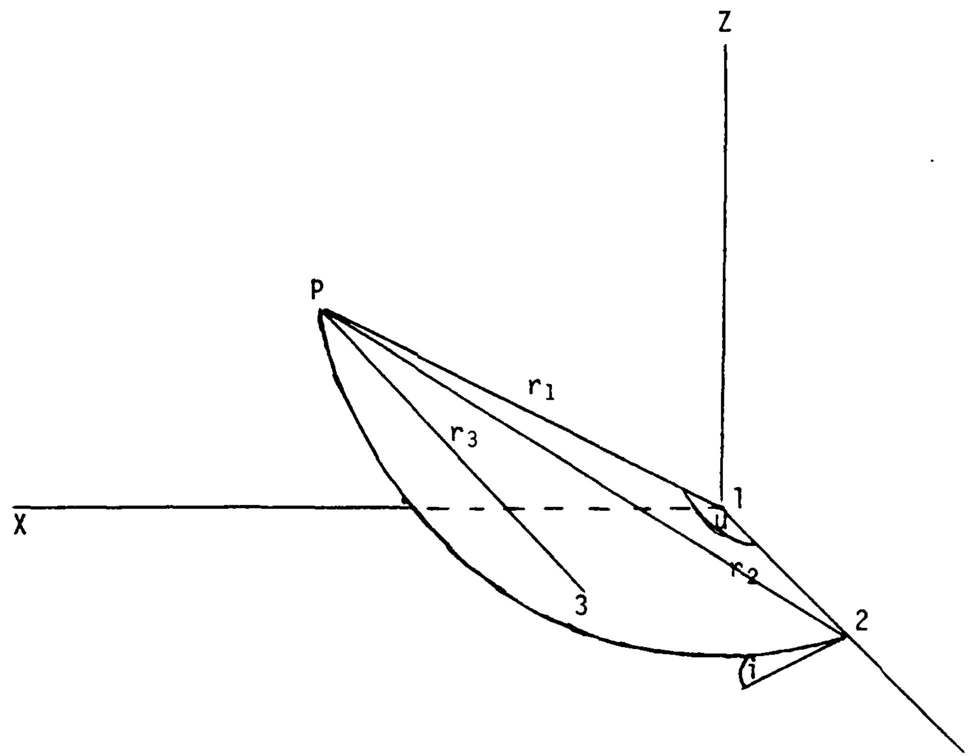

Figure 4

Geometry for Initial Estimate

Writing the coordinates of P we have:

17) $P_x = r_1 \sin u \cos i \quad P_y = r_1 \cos u \quad P_z = r_1 \sin u \sin i$

and with C_x, C_y, C_z as the coordinates of observer 3 then

18) $r_3^2 = (r_1 \sin u \cos i - C_x)^2 + (r_1 \cos u - C_y)^2 + (r_1 \sin u \sin i - C_z)^2$

After squarring terms out, and letting S_3 represent the distance of observer 3 from the origin so that

$$19)\quad S_3^2 = C_x^2 + C_y^2 + C_z^2$$

we get

$$20)\quad r_3^2 = r_1^2 + S_3^2 - 2r_1 C_y \cos u$$
$$-2r_1 \sin u \sqrt{C_x^2 + C_z^2}\left[\frac{C_x \cos i}{\sqrt{C_x^2 + C_z^2}} + \frac{C_z \sin i}{\sqrt{C_x^2 + C_z^2}}\right]$$

Defining the quantities

$$21)\quad \cos \tau = \frac{C_x}{\sqrt{C_x^2 + C_z^2}} \qquad \sin \tau = \frac{-C_z}{\sqrt{C_x^2 + C_z^2}}$$

and using a trigonometric identity, we solve for the angle $i+\tau$ from 20)

$$22)\quad \cos (i+\tau) = \frac{-r_3^2 + r_1^2 + S_3^2 - 2r_1 C_y \cos u}{2r_1 \sin u \sqrt{C_x^2 + C_z^2}}$$

We now recognize that τ is a small angle(assuming $C \neq 0$) since it measures how far observer 3 is out of the x-y plane and all observers will nominally be positioned in the x-y plane. Thus the quadrant of $i+\tau$ is the same as the quadrant of i. Determining i from 21), 22) and the remark following 22) and multiplying by minus 1 which is consistent with P below the x-y plane, - we complete the determination of i. Then use 17) to solve for the coordinates of P.

Numerical Results

In order to illustrate the effect of the geometry on the resultant accuracy in determining the phone position, a number of computer runs were

made with noise added on to the exact ranges. The nominal range was 100 feet and the noise was generated from a uniform random number generator with mean of zero and maximum noise magnitude of .1 ft (which is representative of the equipment being used). Using the ranges, from the three sightings, each run had different angles between their respective lines of sight to the phone.

We give the results as errors Δx, Δy, Δz in determining the various components x,y,z of hydrophone position. In addition we also list the square root of the sum of the squares of the component errors and call this $||\Delta\vec{P}||$. Finally, each run is actually the result of four sub-runs each with a different noise sample. The results listed are for the sub-run with the largest value of $||\Delta\vec{P}||$.

Table 1

Range = 100 Feet

Angle between Lines of Sight	Δx(feet)	Δy(feet)	Δz(feet)	$\lVert\Delta\vec{P}\rVert = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}$ (fe
15°	.21	.33	.03	.40
30°	.11	.17	.03	.20
45°	.07	.11	.03	.14
60°	.06	.09	.03	.11
90°	.04	.06	.04	.09
110°	.03	.05	.08	.11

The above table bears out the theoretical result obtained previously - namely that the errors in locating the hydrophone are minimized when the lines of sight from the observers to the hydrophone are mutually perpendicular.

The method considered here is invariant to range changes and this is shown by running a case with the ranges at 1,000 feet.

Table 2

Range = 1000 feet

Angle between Lines of Sight	Δx(feet)	Δy(feet)	Δz(feet)	$\lVert \Delta\vec{P} \rVert = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}$ (feet)
90°	.04	.06	.04	.09

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